Exercise 69

Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the hyperbola $x^2/A^2 - y^2/B^2 = 1$ are orthogonal trajectories if $A^2 < a^2$ and $a^2 - b^2 = A^2 + B^2$ (so the ellipse and hyperbola have the same foci).

Solution

The points of intersection are found by solving the system of equations for x and y.

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \\ \\ \dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1 \end{cases}$$

Subtract the respective sides of these equations.

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - \left(\frac{x^2}{A^2} - \frac{y^2}{B^2}\right) = 0$$

$$x^2\left(\frac{1}{a^2} - \frac{1}{A^2}\right) + y^2\left(\frac{1}{b^2} + \frac{1}{B^2}\right) = 0$$

$$x^2\left(\frac{A^2 - a^2}{a^2A^2}\right) + y^2\left(\frac{B^2 + b^2}{b^2B^2}\right) = 0$$

$$y^2\left(\frac{B^2 + b^2}{b^2B^2}\right) = x^2\left(\frac{a^2 - A^2}{a^2A^2}\right)$$

Assume that $a^2 - b^2 = A^2 + B^2$ so that $a^2 - A^2 = B^2 + b^2$.

$$y^2\left(\frac{B^2 + b^2}{b^2B^2}\right) = x^2\left(\frac{B^2 + b^2}{a^2A^2}\right)$$

$$y^2\left(\frac{1}{b^2B^2}\right) = x^2\left(\frac{1}{a^2A^2}\right)$$

$$a^2A^2y^2 = b^2B^2x^2$$

$$\frac{A^2y^2}{B^2x^2} = \frac{b^2}{a^2}$$

Differentiate both sides of the given equations with respect to x.

$$\frac{d}{dx}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \frac{d}{dx}(1) \qquad \frac{d}{dx}\left(\frac{x^2}{A^2} - \frac{y^2}{B^2}\right) = \frac{d}{dx}(1)$$

Use the chain rule to differentiate $y = y(x)$.

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \qquad \frac{2x}{A^2} - \frac{2y}{B^2}\frac{dy}{dx} = 0$$

Solve each equation for dy/dx.

$$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2} \qquad -\frac{2y}{B^2}\frac{dy}{dx} = -\frac{2x}{A^2}$$

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y} \qquad \frac{dy}{dx} = \frac{B^2x}{A^2y}$$

At any point of intersection $\frac{A^2y^2}{B^2x^2} = \frac{b^2}{a^2}$, so the slopes of the tangent lines are as follows.

$$\frac{dy}{dx} = -\left(\frac{A^2y^2}{B^2x^2}\right)\frac{x}{y} \qquad \frac{dy}{dx} = \frac{B^2x}{A^2y}$$

$$\frac{dy}{dx} = -\frac{A^2y}{B^2x} \qquad \frac{dy}{dx} = \frac{B^2x}{A^2y}$$

The slopes are negative reciprocals at the points of intersection; therefore, the familes of curves defined by $x^2/a^2 + y^2/b^2 = 1$ and $x^2/A^2 - y^2/B^2 = 1$ are orthogonal trajectories, assuming that $A^2 < a^2$ and $a^2 - b^2 = A^2 + B^2$.